J. Fluid Mech. (1978), *vol.* 84, *part* 1, *pp.* 1–11
Printed in Great Britain

A study of thermal convection in non-Newtonian fluids

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(Received 22 September 1976)

This study considers steady-state, finite amplitude thermal convection in a non-Newtonian fluid. Pseudoplastic (power-law) fluids are considered for a power-law exponent n in the range $1 \leqslant n \leqslant 9$. Finite-difference solutions are obtained for two-dimensional periodic convective modes in a horizontally infinite fluid layer heated from below. The results show that the patterns of convective motions differ only slightly from those in a fluid of constant viscosity for $n \lesssim 3$ while for $n \gtrsim 3$ significant differences are observed. An average viscosity is introduced which provides a good correlation of heat transfer across the layer with the Rayleigh number for the complete range of n considered.

1. Introduction

A variety of fluids are known to have a nonlinear, pseudoplastic rheology. Among these are complex liquids, solutions and suspensions, and solids deforming by steady-state creep. The flow of ice in glaciers is an example of flow by steady-state creep. Another example is flow of polycrystalline rock in the earth's mantle, the study of which has been the primary motivation for the present investigation.

The theory of plate tectonics (see Le Pichon, Francheteau & Bonin 1973) requires that flow has occurred in the earth's mantle over time scales of tens of millions of years. These convective motions are thought to be generated by the heat produced by the decay of radioactive isotopes, and previous studies of finite amplitude thermal convection (see Turcotte & Oxburgh 1967; McKenzie, Roberts & Weiss 1974) indicate the feasibility of this hypothesis. Thus thermal convection in the earth's mantle is believed to have played an important role in the development of surface geological features and in the long-term thermal and chemical evolution of the earth. To understand these processes better, studies have been made of steady-state, finite amplitude thermal convection in a non-Newtonian fluid having a power-law or pseudoplastic rheology with strain rate proportional to stress to the power n.

Several previous studies of thermal convection in non-Newtonian fluids have been made. Liang & Acrivos (1970) studied thermal convection in polymer solutions experimentally. The viscosity of their polymer solutions decreased with increasing strain rate, approaching uniform values at both high and low strain rates. The maximum variation of the viscosity was slightly more than two orders of magnitude over a strain-rate variation of six orders of magnitude. They concluded from their experiments that convective motions in non-Newtonian fluids differed only slightly from those in fluids of uniform viscosity. However, for creep in the earth's mantle, the effective viscosity is expected to be strain-rate dependent over a much wider range of strain rates and the

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magnitude of a limiting viscosity at low strain rates is uncertain (see Parmentier, Turcotte & Torrance 1976).

Ozoe & Churchill (1972) obtained finite-difference solutions for thermal convection in a layer of power-law fluid. Their solutions were for finite Prandtl number with $1 \leqslant n \leqslant 2$. They attempted to determine a critical Rayleigh number for non-Newtonian fluid layers. However, as recognized by Tien, Tsuei & Sun (1969) and as pointed out by Parmentier *et al.* (1976), the nonlinearity introduced by a power-law rheology makes the marginal-stability problem nonlinear, and a critical Rayleigh number cannot be defined independently of the amplitude and mode shape of the disturbance initiating the motion.

Van der Borght, Murphy & Steiner (1974) have considered finite amplitude thermal convection in a layer of a Maxwellian viscoelastic fluid. Their solutions were obtained numerically by a Galerkin technique. Results for viscoelastic and power-law fluids cannot be directly compared.

The present study considers steady-state thermal convection in a horizontally infinite fluid layer heated from below. The fluid layer is bounded above and below by rigid, no-slip boundaries on which the temperature is prescribed. The governing equations are formulated and solutions are obtained by finite-difference approximations. Very viscous fluids (infinite Prandtl number) with a power-law rheology are considered for $1 \leqslant n \leqslant 9$. A previous paper (Parmentier *et al.* 1976) considered the case $n = 3$ in some detail. These studies showed that the differences in the structure of thermal convection patterns in Newtonian and non-Newtonian fluids with $n \leqslant 3$ were small and that the effects of a non-Newtonian viscosity were consistent with the experimental results of Liang & Acrivos (1970). This paper considers larger values of n, $3 \leqslant n \leqslant 9$, for which significant effects on flow structure are caused by the strain-rate dependence of the viscosity. An average viscosity, defined on the basis of energy considerations, is shown to provide a good correlation of the heat flux with the Rayleigh number based on this average viscosity.

2. Mathematical formulation

The equations for conservation of mass, momentum and energy in a viscous compressible fluid are given by Batchelor (1967). These equations are simplified using the Boussinesq approximation, which is rigorously discussed by Mihaljan (1962). In tensor notation with co-ordinates x_i, velocities u_i, pressure p and deviatoric stress τ_{ij}, the equations for steady-state flow are

$$\partial u_i/\partial x_i = 0, \tag{1}$$

$$\frac{1}{Pr}\left(u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}\tau_{ij} + Ra\theta\gamma_i, \tag{2}$$

$$u_j \partial\theta/\partial x_j = \partial^2\theta/\partial x_j \partial x_j, \tag{3}$$

where γ_i is a unit vector in the direction of gravity. Here p' is the pressure deviation $p - p_0$ from the hydrostatic value defined by

$$\partial p_0/\partial x_i = \rho_0 g \gamma_i,$$

ρ_0 being a reference density of the fluid and g the gravitational acceleration. The density of the fluid is assumed to depend only on temperature with

$$\rho-\rho_0 = -\rho_0\alpha(T-T_0),$$

where α is the coefficient of thermal expansion and T_0 is a reference temperature, taken to be the temperature at the top boundary of the fluid layer.

In these equations, the variables have been non-dimensionalized with a length d, the depth of the convecting layer, a time d^2/κ, where κ is the thermal diffusivity of the fluid, a reference viscosity μ_0 and the reference density ρ_0. The dimensionless temperature is defined as

$$\theta = (T-T_0)/\Delta T, \tag{4}$$

where ΔT is the temperature difference between the top and bottom boundaries of the fluid layer. The Rayleigh number and the Prandtl number

$$Ra = \alpha g \Delta T d^3/\nu_0\kappa, \quad Pr = \nu_0/\kappa \tag{5}$$

appear as dimensionless parameters with $\nu_0 = \mu_0/\rho_0$. All the thermodynamic and transport coefficients of the fluid are assumed to be constant with the exception of the viscosity. The case $Pr \to \infty$ is considered so that the inertia terms in the momentum equation may be neglected.

The most general isotropic constitutive equation for a pseudoplastic power-law fluid may be written (see Serrin 1959, p. 233) as

$$\dot{e}_{ij} = A_1\delta_{ij} + A_2\tau_{ij} + A_3\tau_{ik}\tau_{kj} \tag{6}$$

with strain rate

$$\dot{e}_{ij} = \tfrac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$$

and coefficients A_1, A_2 and A_3 which depend on the three invariants of the deviatoric stress tensor

$$\sigma_1 = \tau_{ii} \equiv 0, \quad \sigma_2 = \tau_{ij}\tau_{ij}, \quad \sigma_3 = \tau_{ij}\tau_{jk}\tau_{ki}$$

and on thermodynamic state variables, i.e. pressure and temperature. For deformations which are incompressible ($\dot{e}_{ii} = 0$), (6) reduces to

$$\dot{e}_{ij} = A_1\delta_{ij} + A_2\tau_{ij} - (3A_1/\sigma_3)\tau_{ik}\tau_{kj}, \tag{7}$$

Without further, more specialized, assumptions about the nature of the fluid, this is the simplest constitutive equation which can describe a pseudoplastic fluid.

In this study the simpler law given in dimensional form by

$$\dot{e}_{ij} = (\tau^{n-1}/A^n)\tau_{ij} \tag{8}$$

is considered, where $\tau = \sigma_2^{\frac{1}{2}}$. This law assumes that $A_1 = 0$ and that $A_2 = 1/A^n$ depends only on σ_2. For this law, each strain-rate component is proportional to the corresponding component of deviatoric stress. Nye (1953) considers (8) adequate to describe the steady-state creep of polycrystalline ice. Stocker & Ashby (1973) and Weertman & Weertman (1975) suggest that it can also describe the deformation of polycrystalline rock.

For the rheology represented by (8), an effective scalar viscosity can be introduced by taking

$$\mu = A^n/2\tau^{n-1}, \tag{9a}$$

or in terms of strain-rate components,

$$\mu = \tfrac{1}{2}A\dot{e}^{1/n-1}, \tag{9b}$$

where $\dot{e} = (\dot{e}_{kl}\dot{e}_{kl})^{\frac{1}{2}}$. The viscosity at the reference strain rate κ/d^2 is chosen as the reference viscosity μ_0 introduced earlier:

$$\mu_0 = \tfrac{1}{2}A(\kappa/d^2)^{1/n-1}.$$

Then the dimensionless viscosity $\pi = \mu/\mu_0$ becomes simply

$$\pi = \dot{e}^{1/n-1} \tag{10}$$

with

$$\tau_{ij} = 2\pi\dot{e}_{ij}, \tag{11}$$

where τ_{ij} and $\dot{e}_{ij}$ have been non-dimensionalized by $\mu_0\kappa/d^2$ and κ/d^2 respectively.

Two-dimensional fluid motions periodic in the horizontal co-ordinate with wavelength λ are considered. For two-dimensional motion, the governing equations can be further simplified by taking $\mathbf{x} = (x, z)$ and $\mathbf{u} = (u, w)$ and by introducing a stream function ψ. The z co-ordinate is vertical and $\boldsymbol{\gamma} = (0, -1)$. The velocity components

$$u = \partial\psi/\partial z, \quad w = -\partial\psi/\partial x \tag{12}$$

identically satisfy the continuity equation (1). Cross-differentiating and subtracting the x and z momentum equations (2) eliminates the pressure and gives

$$\nabla^2(\pi\nabla^2\psi) = Ra\frac{\partial\theta}{\partial x} + 2\left\{\frac{\partial^2\pi}{\partial x^2}\frac{\partial u}{\partial z} - \frac{\partial^2\pi}{\partial z^2}\frac{\partial w}{\partial x} + \frac{\partial^2\pi}{\partial x\,\partial z}\left(\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x}\right)\right\}. \tag{13}$$

For two-dimensional flow, the transport equation for thermal energy becomes

$$\frac{\partial}{\partial x}(u\theta) + \frac{\partial}{\partial z}(w\theta) = \nabla^2\theta, \tag{14}$$

where

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2.$$

The solutions to be considered are periodic in the x co-ordinate with rigid (no-slip) isothermal boundaries at $z = 0$ and $z = 1$ on which the temperature is prescribed to be $\theta = 1$ and $\theta = 0$ respectively. The stream function is zero on both boundaries.

Solutions will be discussed in terms of an average viscosity $\bar{\mu}$ given in dimensionless terms by

$$\bar{\pi} = \int_S \pi\dot{e}^2\,dS \bigg/ \int_S \dot{e}^2\,dS, \tag{15}$$

where S is the domain $0 \leqslant x \leqslant \lambda$, $0 \leqslant z \leqslant 1$ in the x, z plane. Then the Rayleigh number based on the viscosity $\bar{\mu}$, which will be termed the apparent Rayleigh number, is given by

$$\overline{Ra} = Ra/\bar{\pi}.$$

To understand the motivation for defining an average viscosity by (15), a global mechanical energy equation is derived by multiplying the momentum equation (2) by the velocity u_i and integrating over a prescribed region in space, in this case the domain S. For rigid (no-slip) or free (zero-shear-stress) isothermal boundaries, this gives

$$Ra(Nu-1)\lambda = \int_S \tau_{ij}\frac{\partial u_i}{\partial x_j}\,dS, \tag{16}$$

where Nu is the Nusselt number defined in terms of the net heat flux $\dot{Q}$ transported across the fluid layer as

$$Nu = \dot{Q}d/k\Delta T\lambda,$$

where k is the thermal conductivity of the fluid. Noting that τ_{ij} is symmetric, (16) can be written as

$$Ra(Nu-1)\lambda = \int_S \tau_{ij}\dot{e}_{ij}dS. \tag{17}$$

This equation expresses a balance between the rate of work done by the motion against viscous stresses and the rate of release of buoyant energy by the fluid motion. Introducing (11) and the definition of $\overline{Ra}$ reduces (17) to

$$\overline{Ra}(Nu-1)\lambda = \int_S \dot{e}^2 dS. \tag{18}$$

Therefore, when expressed in terms of the apparent Rayleigh number $\overline{Ra}$, the mechanical energy balance has no explicit dependence on the form of the viscosity law and, as will be shown later, an average viscosity defined by (15) provides an empirical basis for correlating the Nusselt number with $\overline{Ra}$ for a wide range of Ra and n.

The equations describing finite amplitude thermal convection, (12)–(14) with the viscosity law given by (10), were solved with finite-difference approximations. Derivatives appearing in the equations were approximated by central-difference derivatives on a uniform spatial mesh of grid points. Convective derivatives in the thermal energy equation were approximated by conserving upwind-difference derivatives. The fourth-order equation (13) was solved as two coupled second-order equations by introducing the vorticity

$$\omega = -\nabla^2\psi. \tag{19}$$

The resulting systems of coupled, nonlinear, algebraic equations were solved by the explicit iterative method discussed by Turcotte, Torrance & Hsui (1973). For the range of n considered the nonlinearity of (13) introduces no numerical instabilities not present in the constant-viscosity problem although iterative convergence of the solutions to a steady state is slower as n increases.

The accuracy of the finite-difference solutions has been studied by comparing solutions on successively refined grids. Such convergence studies are essential for establishing the validity of finite-difference solutions and are particularly important for power-law fluids because the viscosity π is singular at points in the flow where $\dot{e}$ vanishes. Results on grids as fine as 21×21 show that the flow in such regions is adequately resolved in so far as global flow properties are concerned. For a sequence of solutions on 12×12, 15×15 and 21×21 grids, global flow properties such as Nu and $\psi_{\max}$ show $O(\Delta x)^2$ convergence trends. Error estimates are obtained by extrapolating the convergence trends to zero grid spacing. For moderate $\overline{Ra}$ ($\lesssim 10^5$), solutions on 15×15 and 21×21 grids have estimated truncation errors of 12 % and 7 % in Nu respectively. The rate of convergence is nearly identical for constant- and variable-viscosity solutions at the same value of $\overline{Ra}$. Owing to the increasingly slow iterative convergence with increasing n, solutions on 21×21 grids have been obtained only for $n \leqslant 5$. The reason for the slow convergence, although not well understood, is related to the increasing nonlinearity of (13). The solutions on 15×15 grids adequately resolve the important details of the thermal and mechanical structure of the flow although greater accuracy might be desirable if experimental results were available for direct comparison.

Even on much more refined grids, finite-difference solutions cannot resolve details of flow in the immediate vicinity of points of vanishing strain rate, where the viscosity becomes infinite. It is unlikely that any real fluid can be found which exhibits power-law behaviour to vanishingly small strain rates. Nevertheless it is important to understand the possible effect that regions of low strain rate would have on the flow. This has been examined by considering the more general viscosity law

$$\pi = \frac{1}{1/\pi_0 + \dot{e}^{1-1/n}}, \tag{20}$$

which gives $\pi \sim \pi_0$ as $\dot{e} \to 0$. For $\pi_0 \to \infty$ this reduces to the form of (10). Detailed studies of the effect of varying π_0 for $n = 3$ are reported by Parmentier *et al.* (1976). For values of π_0 large compared with the average viscosity $\bar{\pi}$, increasing π_0 further had no significant effect on the solutions, suggesting that regions of locally high viscosity do not have an important effect on the overall flow structure. Results for larger n ($n = 7$ was the largest considered) are similar.

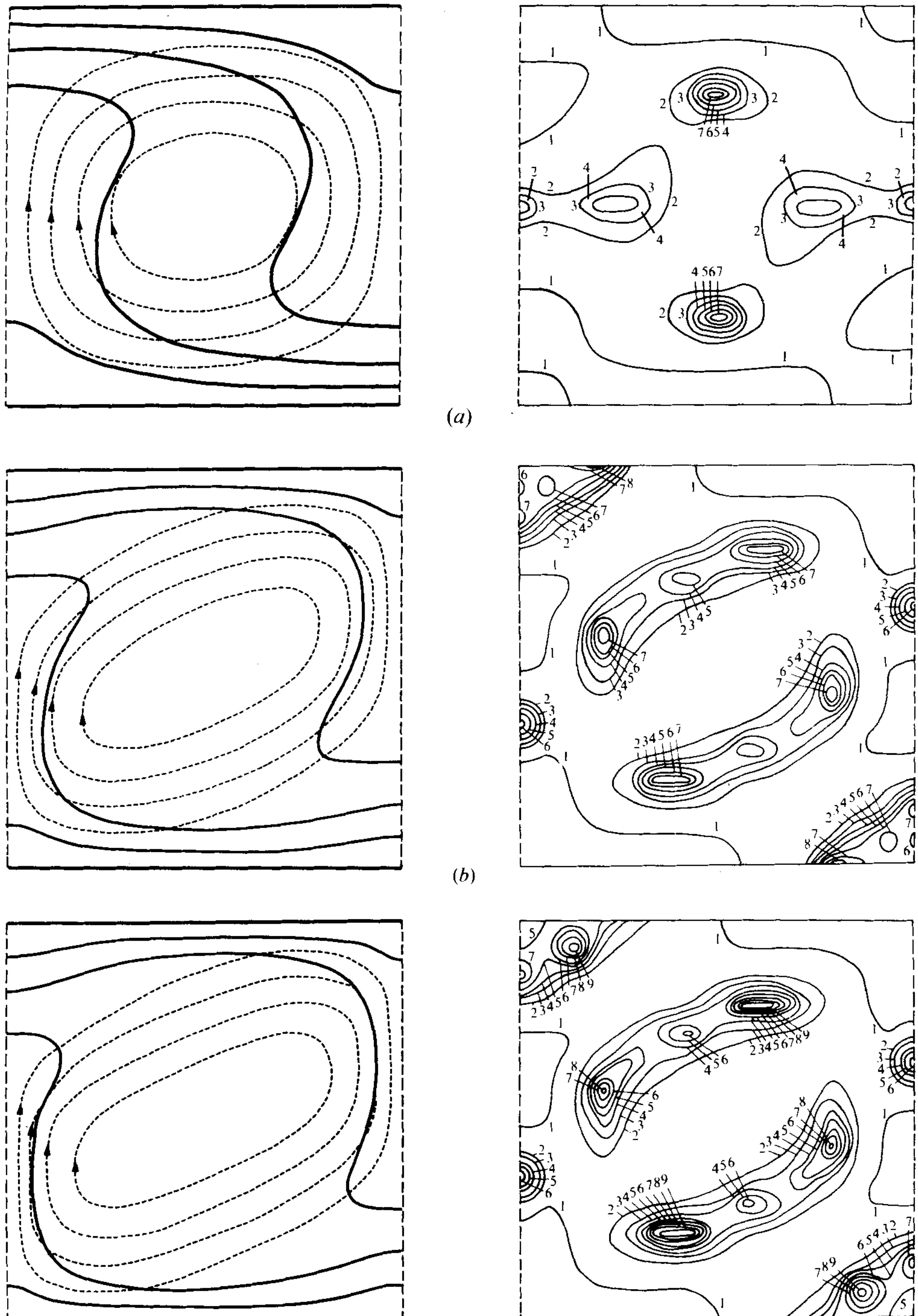
3. Results and discussion

Results have been obtained for a range of Rayleigh numbers and for $1 \leqslant n \leqslant 9$. For all the solutions reported here $\lambda = 2$, i.e. the cell width is equal to the layer depth. Detailed discussion of solutions with $n = 3$ is given by Parmentier *et al.* (1976). In this case the solutions for a non-Newtonian fluid are very similar to those for a constant-viscosity fluid. Attention here is directed to larger values of n, $3 \leqslant n \leqslant 9$, for which the strain-rate dependence of viscosity has a significant effect on the flow structure.

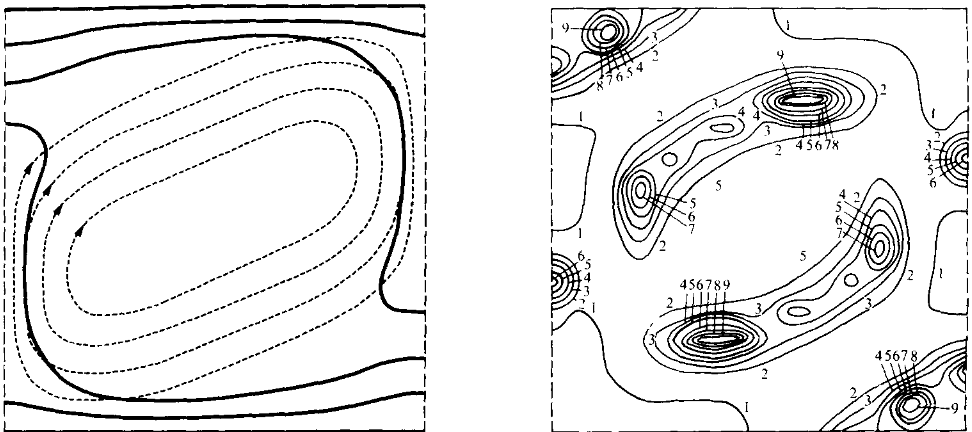
Figure 1 presents a sequence of solutions for $Ra = 500$ with n varying from 3 to 9. For each value of n the structure of the flow in one cell of the periodic pattern is represented in terms of isotherms, given at equal fractions of the temperature difference ΔT between the boundaries, streamlines, given at equal fractions of ψ_{max}, and viscosity contours, given at multiples of the average viscosity $\bar{\pi}$. The values of Nu, ψ_{max} and $\bar{\pi}$ for each value of n are given in the figure caption. For the purpose of comparison, all of these solutions are on 15×15 grids.

The thermal structure remains generally similar for all the solutions, consisting of hot ascending and cold descending plumes in which buoyant potential energy driving the viscous motion is released. The plumes form as continuations of thermal boundary layers adjacent to the top and bottom boundaries of the cell. The interior of the cell is isothermal. For increasing n the thermal boundary layers become thinner and Nu and ψ_{max} increase. However, as will be shown, this represents only an increase in the apparent Rayleigh number.

Important changes in the flow pattern occur with increasing n. As n increases regions of stagnant fluid form above the hot ascending and below the cold descending thermal plumes. In the finite-difference solutions, the fluid in these regions is not completely stagnant but undergoes a very weak recirculation. Corresponding changes can be seen in the viscosity field, the stagnant regions appearing as regions of high viscosity.



FIGURES 1 (*a–c*). For legend see next page.



(*d*)

FIGURE 1. Isotherms, streamlines and contours of constant viscosity for $Ra = 500$. Isotherms (solid) and streamlines (dashed) are given at equal fractions of ΔT and $\psi_{\max}$ respectively. Viscosity contours are given at integer multiples of the average viscosity $\bar{\pi}$ defined in (15). (*a*) $n = 3$ with $Nu = 3{\cdot}16$, $\psi_{\max} = 11{\cdot}2$ and $\bar{\pi} = 3{\cdot}81 \times 10^{-2}$. (*b*) $n = 5$ with $Nu = 4{\cdot}43$, $\psi_{\max} = 27{\cdot}5$ and $\bar{\pi} = 7{\cdot}76 \times 10^{-3}$. (*c*) $n = 7$ with $Nu = 5{\cdot}36$, $\psi_{\max} = 39{\cdot}7$ and $\bar{\pi} = 3{\cdot}75 \times 10^{-3}$. (*d*) $n = 9$ with $Nu = 6{\cdot}12$, $\psi_{\max} = 49{\cdot}9$ and $\bar{\pi} = 2{\cdot}35 \times 10^{-3}$.

The viscous dissipation rate, given by

$$\phi = \tfrac{1}{2}\tau_{ij}\dot{e}_{ij} = \pi^{-(2n+1)/(n-1)}, \tag{21}$$

varies inversely with the viscosity π for $n > 1$. Therefore the stagnant regions of high viscosity are regions of low viscous dissipation. The last expression in (21) follows from (10) and (11). An indeterminate form results for $n = 1$ since $\pi = 1$. In this case the viscous dissipation cannot be expressed as a function of the viscosity alone.

For all values of n, high viscosity occurs in isolated regions where the strain rate vanishes. Two such regions occur on the planes $x = 0$ and $x = \frac{1}{2}\lambda$ but appear to have no effect on the flow since stresses are small in these regions. Broad arcuate zones of high viscosity containing regions of locally higher viscosity occur in the interior of the flow. However, the viscosity in regions of high strain rate is approximately equal to the average value $\bar{\pi}$ for the complete range of n considered. This is an indication that the viscous dissipation required to balance the buoyant potential energy released by the motion occurs in regions of high strain rate.

As stated earlier, the apparent Rayleigh number based on the average viscosity $\bar{\pi}$ provides a means of correlating the Nusselt number for various values of n. The Rayleigh number $\overline{Ra}$ based on the average viscosity has been calculated for each case shown in figure 1. The Nusselt number is plotted as a function of $\overline{Ra}$ in figure 2. Also plotted in this figure is the Nusselt number for constant-viscosity fluids ($n = 1$ and $Ra = \overline{Ra}$) over a range of $\overline{Ra}$. An excellent correlation is noted between results for non-Newtonian and constant-viscosity fluids. In both cases Nu varies approximately as $\overline{Ra}^{\frac{1}{3}}$, the variation found from the experimental results of Silveston (1958) for a constant-viscosity fluid. Since $\overline{Ra}$ increases with increasing n at constant Ra, this gives rise to the increase in apparent Rayleigh number noted earlier in discussing the results in figure 1.

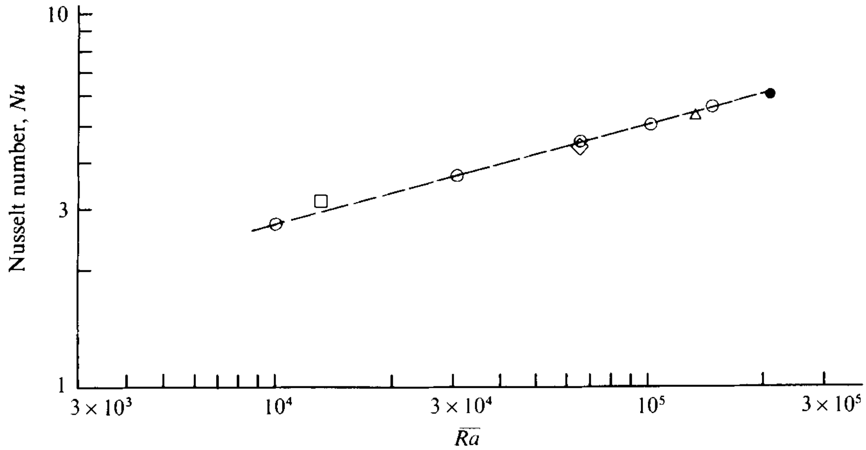


FIGURE 2. Nusselt number as a function of apparent Rayleigh number $\overline{Ra}$, showing correlation of results for constant viscosity with those for power-law fluids for a range of n. ○, $n = 1$, $Ra = \overline{Ra}$; □, $n = 3$, $Ra = 500$; ◇, $n = 5$, $Ra = 500$; △, $n = 7$, $Ra = 500$; ●, $n = 7$, $Ra = 500$.

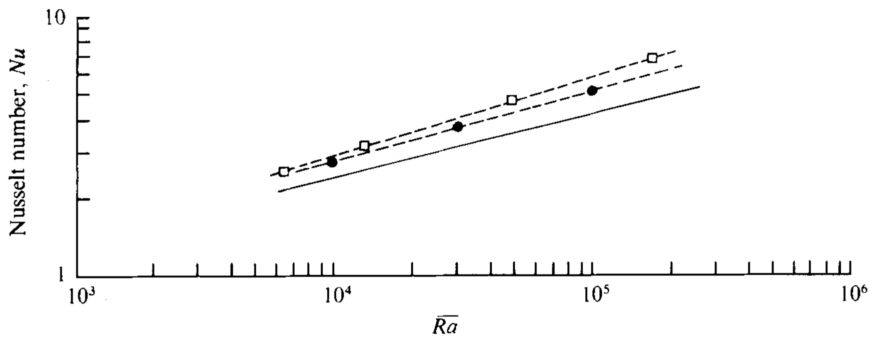


FIGURE 3. Nusselt number as a function of apparent Rayleigh number $\overline{Ra}$, showing correlation of results for constant viscosity (circles) with those for a power-law fluid with $n = 3$ (squares) for a range of $\overline{Ra}$ (from Parmentier *et al.* 1976). Also shown (solid line) is the experimental correlation $Nu = 0{\cdot}24\,\overline{Ra}^{\frac{1}{4}}$ of Silveston (1958).

The correlation of the Nusselt number with $\overline{Ra}$ also holds for $n = 3$ over a wide range of Rayleigh numbers. This is shown by the results in figure 3, taken from Parmentier *et al.* (1976). Also shown in this figure is the correlation of Silveston (1958). It is important to note that the correlation of Nu with $\overline{Ra}$ is purely empirical and does not follow directly from the energy balance given in (18). That the energy balance, when written in terms of the average viscosity, has no explicit dependence on n only suggests that this correlation might hold.

Cells of unit aspect ratio ($\lambda = 2$) remain stable to two-dimensional disturbances in the x, z plane for the complete range of parameters investigated. Variations of λ were considered only for $n = 3$. In that case, the Nusselt number was a maximum in the neighbourhood of $\lambda = 2$. This does not necessarily mean that the two-dimensional cellular structure is stable to three-dimensional disturbances or that it is a physically realizable flow pattern under all conditions. In a constant-viscosity fluid, the theory of Busse (1967) and the experiments of Krishnamurti (1970) have shown that two-dimensional rolls are stable for only a limited range of Rayleigh numbers above the critical value. Liang & Acrivos (1970) observed that two-dimensional rolls in their non-Newtonian fluid transformed to a more complex three-dimensional pattern at approximately the same value of the Rayleigh number as in constant-viscosity fluids. This may not be surprising since for the conditions of their experiments fluid behaviour would have been only weakly non-Newtonian. Since the present study considers only two-dimensional motion, the range of stability of two-dimensional rolls in a highly non-Newtonian fluid cannot be determined.

4. Conclusions

For weakly non-Newtonian fluids ($n \lesssim 3$) differences in the structure of steady-state thermal convection cells from that for Newtonian fluids is small. This is not true for larger n. As n increases, the flow patterns show that fluid deformation tends to become more localized and regions of stagnant fluid develop.

An average viscosity has been defined, on the basis of energy considerations, as the average over the region of flow of the viscosity weighted by the strain rate squared. A good correlation of the heat flux with a Rayleigh number based on this average viscosity results that is independent of the exponent in the viscosity law. This is true despite significant differences in flow structure.

This research has been partially supported by the Earth Sciences Section of the National Science Foundation under NSF grant DES72-01522. Final stages of the work were completed while the author was at Oxford University supported by a fellowship from the National Science Foundation.

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